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NUTATION-FREE MOTIONS IN A SOLUTION OF THE PROBLEM OF MOTION OF A GYROSTAT

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A solution of the problem of motion of a gyrostat obtained by E. I. Kharlamova in [1] is investigated. The conditions of existence of nutation-free motions (in which the nutation angle is constant) in this solution, are derived. A survey of the basic results obtained in the problem of nutation-free motions is given in [2].

Kharlamova determined a solution based on an integro-differential equation which was also derived by her in [1]. The equation was obtained from the equations given in [3], under the assumption that the center of gravity and the gyrostatic moment vector lie in the principal plane of the inertia ellipsoid constructed for a fixed point. Following [1], we denote by a, a_1, a_2, b_1, b_2 the components of the gyration tensor in the special axes; by $\lambda, \lambda_1, \lambda_2$ the components of the gyrostatic moment; by ν, ν_1, ν_2 the components of the unit vector collinear with the force of gravity; by x, y, z the components of the moment of momentum; by Γ the product of the weight of the gyrostat and the distance between the center of gravity and the fixed point. We adopt the Hesse condition $(a_2 = a_1)$ and $b_2 = 0, \lambda_2 = 0$. Finally we pass to the dimensionless variables and parameters. To do this we refer the variables x, y, z, ξ [1] and the parameters λ, λ_1 to the quantity $\sqrt{\Gamma / b}$, and the components a, a_1 of the gyration tensor to b. Then the Kharlamova solution becomes

$$x = \xi + a_1\lambda_1, \quad y = \frac{c_0}{\xi} + l + \xi\left(c_2 - \frac{a - a_1}{2}\right) \tag{1}$$

$$z^2 = \frac{1}{\xi^2}(m_4\xi^4 + m_3\xi^3 + m_2\xi^2 + m_1\xi - c_0^2)$$

$$\nu = s_0 + s_1\xi + s_2\xi^2, \quad \nu_1 = \frac{c_0c_1}{\xi} + s_0' + s_1'\xi + s_2'\xi^2$$

$$\nu_2 = z(c_1 + 2c_2\xi), \quad \frac{d\xi}{dt} = -\xi z$$

where

$$l = c_1 + \frac{a_1(a\mu - \lambda^*)}{aa_1 - 1}, \quad s_0 = \frac{1}{a_1}(c_1^2 + a_1c_1\lambda^* + 2c_0c_2) \tag{2}$$

$$s_0' = c_1^2 + c_1\mu + 2c_0c_2, \quad s_1 = \frac{1}{a_1^2 + 1}[6a_1c_1c_2 + c_1(a_1^2 + 1) + 2a_1c_2(\mu + a_1\lambda^*)]$$

$$s_1' = \frac{1}{2(a_1^2 + 1)}[6c_1c_2(a_1^2 - 1) - c_1(a - a_1)(a_1^2 + 1) + 4a_1c_2(a_1\mu - \lambda^*)]$$

$$s_2 = \frac{c_2}{4 + a_1^2}[6a_1c_2 + (3a_1^2 - aa_1 + 4)], \quad s_2' = \frac{c_2}{4 + a_1^2}[2(a_1^2 - 2)c_2 -$$

$$(2a + aa_1^2 - a_1^3)]$$

$$m_1 = -2c_0(c_1 + \mu)$$

$$m_2 = -\frac{c_0}{a_1(4 + a_1^2)}[4 - a_1(a - a_1)(3 + a_1^2) + 2a_1c_2(1 + a_1^2)] -$$

$$\frac{1}{2a_1c_2(1 + a_1^2)}[c_1^2(1 + a_1^2) + 4a_1^2c_1c_2(2\lambda^* + a_1\mu) +$$

$$2a_1^3c_2(\mu^2 + \lambda^{*2}) + 2a_1c_1^2c_2(4 + a_1^2)]$$

$$m_3 = \frac{1}{a_1^2 + 1}\{2c_1c_2(5 - a_1^2) + c_1(a - a_1)(1 + a_1^2) +$$

$$2c_2[2a_1\lambda^* + \mu(1 - a_1^2)] - (a_1^2 + 1)[2\lambda^* - \mu(a - a_1)]\}$$

$$m_4 = \frac{1}{4(4 + a_1^2)}\{4(8 - a_1^2)c_2^2 + 4[2a + a_1^2(a - a_1)]c_2 -$$

$$(4 + a_1^2)[4 + (a - a_1)^2]\}$$

$$c_1 = \frac{1}{(4 + a_1^2)[6c_2 + (a + a_1)]}\{2c_2[\mu(a_1^2 - 2) - a_1\lambda^*(2a_1^2 + 5)] +$$

$$\lambda^*(3a_1^2 - aa_1 + 4) + \mu(a_1^3 - a_1^2a - 2a)\}$$

$$6c_2 = 2\delta - (a + a_1), \quad \delta = \pm(a^2 + a_1^2 - aa_1 + 3)^{1/2}$$

$$\lambda^* = \lambda + a_1\lambda_1, \quad \mu = a_1\lambda + \lambda_1(a_1^2 - aa_1 + 1)$$

The quantity c_0 is given by

$$Ac_0^2 + Bc_0 + C = 0 \tag{3}$$

$$A = 8c_2^3(1 + a_1^2)(4 + a_1^2), \quad B = 2c_1c_2\ \{2c_1c_2(8 - 11a_1^2 - a_1^4) +$$

$$4a_1c_2(4 + a_1^2)(\lambda^* - a_1\mu) + a_1c_1(1 + a_1^2)[a_1(a_1 - a) - 4]\}$$

$$C = (4 + a_1^2)\{c_1^4[2c_2(1 - 2a_1^2) - a_1(1 + a_1^2)] + 4a_1c_1^3c_2[\lambda^*(1 - a_1^2) +$$

$$a_1\mu] + 2a_1^2c_1^2c_2(\lambda^{*2} + \mu^2) - 2a_1^2c_2(1 + a_1^2)\}$$

In contrast to [1], the dependence of the quantities m_2 and c_0 on the basic parameters is given here in the explicit form.

We note that $\lambda^* = 0, \mu = 0$ yields a solution obtained by Dokshevich in [4] which was given a geometrical interpretation in [5] using the hodograph method.

Let us consider the domain of variation of the dimensionless parameters in which the solution is real. Since we investigate the problem of motion of a gyrostat, the triangular inequalities imposed on the moments of inertia of the gyrostat are discarded. From the conditions of positive definiteness of the kinetic energy of the gyrostat, follows $aa_1 - 1 > 0$. The second restriction imposed on the parameters is given by Eq. (3): $B^2 - 4AC \geqslant 0$. The variable ξ varies over the interval where the right-hand side of the expression for z^2 in (1) is nonnegative.

Let us find the limiting value of the function z^2

$$m_4\xi^4 + m_3\xi^3 + m_2\xi^2 + m_1\xi - c_0^2 = 0$$

The discriminant of Eq. (4) has the form

$$G = g_2^3 - 27g_3^2$$

$$g_2 = -m_4c_0^2 - {}^1/_4 m_1m_3 + {}^1/_{12} m_2^2$$
$$g_3 = -{}^1/_{16} m_2m_4c_0^2 + {}^1/_{48} m_1m_2m_3 - {}^1/_{16} m_1^2m_4 + {}^1/_{16} m_3^2c_0^2 - {}^1/_{216} m_2^3$$

We write the inequalities

$$ {}^1/_{16} m_3^2 - {}^1/_6\, m_2m_4 > 0, \quad {}^3/_{16} m_3^4 - m_2m_3^2m_4 + m_2^2m_4^2 + 4m_4^3c_0^2 + m_1m_3m_4^2 > 0 \qquad (5)$$

which determine, together with the discriminant, the condition for the roots of (4) to be real. The conditions imposed on the parameters under which the solution will be real are: (1) $G \geqslant 0$ and the inequalities (5), (2) $G < 0$.

Let us consider the conditions of existence of nutation-free motions relative to the vertical in the solution in question. The equality $\alpha\nu + \beta\nu_1 + \gamma\nu_2 = \alpha_0$, where α, β, γ, α_0 are constants, represents the necessary and sufficient condition for such motions to exist. In the present case we have a constant angle between the vectors $\mathbf{v}$, (ν, ν_1, ν_2) and $\mathbf{e}$ (α, β, γ) ; this angle is permanently tied to the body. Let us substitute ν, ν_1, ν_2 from (1) into the last relation and require that the resulting equality is an identity in ξ. Using the inequality

$$m_4 = -\frac{1}{4 + a_1^2}\left[(a + c_1')^2 + (1 + a_1c_2')^2\right] < 0, \quad c_2' = c_2 - \frac{a - a_1}{2}$$

we find

$$\gamma = 0, \quad c_0c_1 = 0, \quad \alpha s_2 + \beta s_2' = 0, \quad \alpha s_1 + \beta s_1' = 0, \quad \alpha_0 = \alpha s_0 + \beta s_0' \qquad (6)$$

Consider the condition which follows from the third and fourth equations of (6)

$$s_1's_2 - s_1s_2' = 0 \qquad (7)$$

Substituting into it s_1, s_2, s_1', s_2' from (2), we obtain

$$2c_2\,(a_1^2 + 1)\,(a_1\lambda^* - 2\mu) - 2\mu\,(a_1^3 + a_1 + a) + \lambda^*\,(a_1^4 - a_1^3a + 3a_1^2 -$$
$$3aa_1 + 4) = 0$$

or, after some transformations

$$2c_2\,[3a_1\tau - \sigma\,(a_1^2 - 2)] + \sigma\,(2a + aa_1^2 - a_1^3) + \tau\,(3a_1^2 - aa_1 + 4) = 0 \qquad (8)$$
$$\sigma = \mu + a_1\lambda^*, \quad \tau = a_1\mu - \lambda^*$$

It can easily be shown that the solution τ / σ of Eq.(8) can be written in the form $2\tau / \sigma = (a_1 - a) - 2c_2$ which after substitution of c_2 from (2), yields

$$3\tau / \sigma = (2a_1 - a) - \delta \qquad (9)$$

At this particular value of τ / σ we have $c_1 = 0$. This shows that the condition (7) implies that the coefficient c_1 is zero, therefore the equation $c_0c_1 = 0$ holds.

From this it follows that the necessary and sufficient condition of existence of nutation-free motions in the solution in question is given by the relation (9) only.

We shall now show that when condition (9) holds, values of the dimensionless parameters a, a_1, σ, τ exist for which the solution (2) is real. Let $a = 2$, $a_1 = 2$, $\sigma = 4$, $\delta = -2.65$. Then $c_2 = -1.55$, $c_0 = -0{,}64$, $g_2 = 3.38$, $g_3 = 1.21$. For these values of the parameters $G > 0$ and the inequalities (5) hold, consequently Eq. (4) has four real roots. This shows that a nutation-free motion is physically realizable.

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